Lec 33

Surf integral of vec field

Let $\vec{F} = \langle P, Q, R \rangle$ be a vec field in $\mathbb{R}^3$.
S be a parametrised orientable and piecewise smooth surf.
We want to measure the flow of $\vec{F}$ across S.

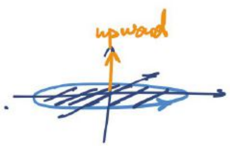
Def S is orientable if it has a cont. choice of normal vec at each point
So there's a well defined inside/bottom and outside/top

Non-example : Mobius strip , Klein bottle ...

The orientation of $\vec{r}(u,v) = \langle f(u,v), g(u,v), h(u,v) \rangle$, we take the orientation as:
$\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ ← points outward if S closed, else we treat as upward

Boundry of surfs and induced orientation

Let S be oriented surf with boundary Ex.



Def We take the boundary to go in counterclockwise direction.
Think induction
→ Try walking along boundary s.t. surf is always to left

Integrate

Def The surf integral of $\vec{F}$ over S is
$\iint_S \vec{F} \cdot \vec{N} \, dS$ where $\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$

Recall for over scalar field

$$\iint_S f\, dS = \iint_D f(\vec{r}(u,v))\, \|\vec{r}_u \times \vec{r}_v\|\, dA$$

For vec field:

$$\iint_S \vec{F}\cdot\vec{N}\, dS = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}\, \|\vec{r}_u \times \vec{r}_v\|\, dA$$

$$= \iint_D \vec{F}\cdot \underbrace{(\vec{r}_u \times \vec{r}_v)}_{d\vec{S}}\, dA$$

$$= \iint_D \vec{F}\cdot d\vec{S}$$

Ex. $\vec{F} = \langle x, y, 0\rangle$ $\quad S$ = bottom half of unit sphere

Parametrise S as $\vec{r}(u,v) = \langle u, v, -\sqrt{1-u^2-v^2}\rangle$ for $(u,v) \in D = \{(u,v) \mid u^2+v^2 \le 1\}$

$$\vec{r}_u = \langle 1, 0, \frac{-u}{\sqrt{1-u^2-v^2}}\rangle$$

$$\vec{r}_v = \langle 0, 1, \frac{-v}{\sqrt{1-u^2-v^2}}\rangle$$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \cdots$$

Then eval $\iint_D \langle u, v, 0\rangle \cdot \vec{N}\, dA$ $\quad \cdots = \frac{4\pi}{3}$

Trick to do triple scalar prod $\vec{a}\cdot(\vec{b}\times\vec{c})$

$$\begin{aligned}\vec{a} &= \langle a_1, a_2, a_3\rangle\\ \vec{b} &= \langle b_1, b_2, b_3\rangle\\ \vec{c} &= \langle c_1, c_2, c_3\rangle\end{aligned} \quad \Rightarrow \quad \vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$